Solution *by Steven Jang, Cal Poly Pomona Problem Solving Group, Pomona, CA*

Problem 885. If $a, b, x, y > 0$ and $n \in \mathbb{N}^*$, prove that

$$\frac{(x+y)^n}{2^{(n-1)}} \leq \frac{(ax+by)^n + (bx+ay)^n}{(a+b)^n} \leq x^n + y^n. \tag{*}$$

Consider the left side of the inequality

$$\begin{aligned}\frac{(x+y)^n}{2^{(n-1)}} &\leq \frac{(ax+by)^n + (bx+ay)^n}{(a+b)^n} \\ \frac{(x+y)^n(a+b)^n}{2^n} &\leq \frac{(ax+by)^n + (bx+ay)^n}{2}\end{aligned}$$

Since the geometric mean is always less than or equal to the arithemtic mean,

$$\sqrt{(ax+by)^n(bx+ay)^n} \leq \frac{(ax+by)^n + (bx+ay)^n}{2} \tag{**}$$

We will show

$$\frac{(x+y)^n(a+b)^n}{2^n} \leq \sqrt{(ax+by)^n(bx+ay)^n}$$

Raising both sides to power $\frac{2}{n}$, this is equivalent to

$$\begin{aligned}\frac{(x+y)^2(a+b)^2}{4} &\leq (ax+by)(bx+ay) \\ (x+y)^2(a+b)^2 &\leq 4ab((x+y)^2 - 2xy) + 4xy((a+b)^2 - 2ab) \\ 0 &\leq 4ab(x+y)^2 + 4xy(a+b)^2 - 16abxy - (a+b)^2(x+y)^2 \\ 0 &\leq 4ab(x-y)^2 - (a+b)^2(x-y)^2 \\ 0 &\leq (x-y)^2(a-b)^2\end{aligned}$$

Since inequality (**) holds true, we can deduce that the left side of inequality (*) is true.

Now, consider the right side of inequality (*)

$$\begin{aligned}\frac{(ax+by)^n + (bx+ay)^n}{(a+b)^n} &\leq x^n + y^n. \\ (ax+by)^n + (bx+ay)^n &\leq (x^n + y^n)(a+b)^n.\end{aligned}$$

Without loss of generality, we may assume that $x > y$. Let $z = \frac{x}{y}$ and $c = \frac{a}{b}$. Dividing both sides of the inequality by $y^n b^n$, we can rewrite the given inequality as

$$(cz+1)^n + (z+c)^n \leq z^n(c+1)^n + (c+1)^n$$

$$\underbrace{\sum_{k=0}^{n} \binom{n}{k} (cz)^{n-k}}_{s_1} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} c^{n-k} z^k}_{s_2} \leq \underbrace{\sum_{k=0}^{n} \binom{n}{k} c^{n-k} z^n}_{s_3} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} c^{n-k}}_{s_4}$$

Notice that the first and last terms of s_1, s_2, s_3, and s_4 cancel out. Comparing the corresponding coefficients of c^{n-p} on each side, we see that the inequality is true if $z^n + 1 > z^{n-p} + z^p$, for every $p = 0, 1, ..., n$.

Let $f(p) = z^{n-p} + z^p$. To find the maximum of the $f(p)$ in the interval $[0, n]$, we must compare the values at the function at critical numbers inside the interval and at the ends of the interval $p = 0$, and $p = n$.

$$\begin{aligned} f'(p) &= -z^{n-p}\log z + z^p \log z \\ 0 &= -z^{n-p} + z^p \\ z^{n-p} &= z^p \\ p &= \frac{n}{2} \end{aligned}$$

The values of $f(p)$ at the three points are

$$\begin{aligned} f(0) &= z^n + 1 \\ f(n) &= z^n + 1 \\ f(\frac{n}{2}) &= 2z^{\frac{n}{2}} \end{aligned}$$

Since the arithmetic mean is always greater than or equal to the geometric mean, we know that

$$\begin{aligned} \frac{z^n + 1}{2} &\geq z^{\frac{n}{2}} \\ z^n + 1 &\geq 2z^{\frac{n}{2}} \end{aligned}$$

Thus, we know that $f(p)$ will never exceed $z^n + 1$ and the right side of inequality (*) holds true.