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Problem 881. Find a formula (possibly recursive) for the number of integers with n digits that contain exactly one 47 in the integer.

Proof.
Let $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the function for the number of integers with n digits that contain exactly one 47 in the integer.

Let $g : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the function for the number of integers with n digits with no 47 in the integer.

To find a recurrence relation of f, we can use the inclusion/exclusion strategy. $f(n-1)$ is the number of integers with $n-1$ digits that contain exactly one 47. By appending one more digit at the end of each of those numbers, we can see that there are $10f(n-1)$ possible integers.

Notice that we need to consider the numbers where the last digit of the numbers with $n-1$ digits end with a 4, since if we append a 7 at the end, it will result in a duplicate 47. We can exclude those values by subtracting $f(n-2)$.

Finally, we need to include the integers where the only 47 are the last two digits, which is equivalent to $g(n-2)$.

We can also use the inclusion/exclusion strategy to find the recurrence relation of g. $g(n-1)$ is the number of integers with $n-1$ digits that do not contain 47. If we append a digit at the end, we can see that there are $10g(n-1)$ possible integers. From the set of $g(n-1)$ integers, we must exclude those that end in a 4 since we must consider the case of appending 7. There are exactly $g(n-2)$ integers that end in a 4.

The functions f and g can be defined by the following recurrence relation

$$f(n) = 10f(n-1) - f(n-2) + g(n-2)$$
$$g(n) = 10g(n-1) - g(n-2)$$

Note that the base cases of f are $f(1) = 0$ and $f(2) = 1$ and the base cases of g are $g(0) = 1$ and $g(1) = 9$

To find the closed form for this recurrence relation, we must first find the roots of the following equation

$$r^2 - 10r + 1 = 0 \implies r = 5 \pm 2\sqrt{6}$$

Then, let us define the closed form of $g(n)$ with the following equation.

$$g(n) = \alpha(5 + 2\sqrt{6})^n + \beta(5 - 2\sqrt{6})^n$$

We can find the constants α and β, by checking the base cases of $g(n)$.

$$\begin{aligned}
g(0) &= 1 = \alpha + \beta \\
g(1) &= 9 = 5\alpha + 5\beta + 2\alpha\sqrt{6} - 2\beta\sqrt{6} \\
\alpha &= \frac{1}{2} + \frac{1}{\sqrt{6}} \\
\beta &= \frac{1}{2} - \frac{1}{\sqrt{6}} \\
g(n) &= \left(\frac{1}{2} + \frac{1}{\sqrt{6}}\right)(5+2\sqrt{6})^n + \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)(5-2\sqrt{6})^n
\end{aligned}$$

Therefore, we can represent the function f as the following non-homogeneous recurrence relation

$$f(n) = 10f(n-1) - f(n-2) + \left(\frac{1}{2} + \frac{1}{\sqrt{6}}\right)(5+2\sqrt{6})^n + \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)(5-2\sqrt{6})^n$$

Note that the characteristic equation for the recurrence relation of $f(n)$ is the same as the characteristic equation for the recurrence relation of $g(n)$. Therefore, the solution is of the form

$$f(n) = A(5+2\sqrt{6})^n + B(5-2\sqrt{6})^n + Cn(5+2\sqrt{6})^n + Dn(5+2\sqrt{6})^n$$

for some $A, B, C, D \in \mathbb{R}$

A, B, C, D can be found by menial computation.

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